Small deviation probabilities for Matérn processes under weighted L_2-norm

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Small deviations

Let $X(t)$, $0 \leq t \leq 1$, be a zero mean Gaussian process with covariance function $G(t,s) = \mathsf{E}X(t)X(s)$ and let ρ be a nonnegative function on $[0,1]$. Denote

$$\|X\|_\rho = \left(\int_0^1 X^2(t)\rho(t)dt \right)^{1/2}.$$

The problem is to describe the behavior of $\mathsf{P}\{\|X\|_\rho \leq \varepsilon\}$ as $\varepsilon \to 0$.

Motivation

Small deviation probabilities are connected with

- metric entropy (J. Kuelbs, W. Li, 1993),
- approximation of stochastic processes (W. Li, W. Linde, 1999),
- functional data analysis (F. Ferraty, Ph. Vieu, Nonparametric Functional Data Analysis, Springer, 2006),
- etc.

Small deviations

The problem of small deviation asymptotics was solved by Sytaya (1974), but in an implicit way. Later Zolotarev; Dudley, Hoffmann-Jørgensen and Shepp; Ibragimov; W. Li; Dunker, Lifshits and Linde; Nazarov and Nikitin, etc. improved and simplified the expression for $\mathsf{P}\{\|X\|_\rho \le \varepsilon\}$.

Karhunen-Loève expansion

By Karhunen-Loève expansion, one has the equality in distribution

$$\|X\|_\rho^2 = \sum_{k=1}^{\infty} \lambda_k \xi_k^2,$$

where ξ_k are i.i.d. $N(0,1)$ r.v.'s,
$\lambda_k > 0$ are the eigenvalues of the integral equation

$$\lambda f(t) = \int_0^1 G(t,s)\sqrt{\rho(t)\rho(s)} f(s) ds, \quad t \in [0,1].$$

Comparison theorem

W. Li, 1992; F. Gao, J. Hannig and F. Torcaso, 2003:

Let ξ_k be i.i.d. $N(0,1)$ r.v.'s, and let $a_k > 0$ and $b_k > 0$ be such that $\prod_{k=1}^{\infty}(b_k/a_k) < \infty$. Then, as $\varepsilon \to 0$,

$$\mathrm{P}\left\{\sum_{k=1}^{\infty} a_k \xi_k^2 \le \varepsilon^2\right\} \sim \left(\prod_{k=1}^{\infty} \frac{b_k}{a_k}\right)^{1/2} \mathrm{P}\left\{\sum_{k=1}^{\infty} b_k \xi_k^2 \le \varepsilon^2\right\}.$$

Exact and logarithmic asymptotics

Exact asymptotics:

$$\mathsf{P}\{\|X\|_\rho \le \varepsilon\} \sim C\varepsilon^{\gamma}\exp(-K\varepsilon^{-\alpha}).$$

Logarithmic asymptotics:

$$\ln \mathsf{P}\{\|X\|_\rho \le \varepsilon\} \sim -K\varepsilon^{-\alpha}.$$

If $\lambda_k = \left(\vartheta(k+\delta+O(k^{-1}))\right)^{-d}$, then

$$\alpha = \alpha(d), \quad K = K(d,\vartheta),$$

$$\gamma = \gamma(d,\delta), \quad C = C(\vartheta, d, \delta)\cdot C_{\mathrm{dist}}(\lambda_k, k\in\mathbb{N}).$$

Exact and logarithmic asymptotics

$$C_{\mathrm{dist}} = \left(\prod_{k=1}^{\infty} \frac{(\vartheta(k+\delta))^{-d}}{\lambda_k} \right)^{1/2}.$$

When the eigenfunctions of the covariance can be expressed in terms of elementary or special functions, there exist exlicit formulas for the distortion constants.

Matérn processes

For $\nu > 1/2$ define the Matérn process $X^{(\nu)}(t)$, $t \in [0,1]$, as a Gaussian process with mean zero and covariance function

$$G_\nu(t,s) = \frac{2^{3/2-\nu}}{\Gamma(\nu - 1/2)} |t-s|^{\nu-1/2} K_{\nu-1/2}(|t-s|),$$

where K_α is the modified Bessel function of order α.

Matérn processes

Matérn processes arise in

- geostatistics (B. Matérn, Spatial Variation, Springer, 1986),
- applied models of statistical fluid mechanics and theory of electrical noise (A. M. Yaglom, Correlation Theory of Stationary and Related Random Functions, Springer, 1987),
- etc.

Logarithmic asymptotics

Denote $J_h = \int_0^1 \rho(t)^{1/h} dt$. Let ρ be a summable nonnegative function on $[0, 1]$. Then

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(2\nu-1)} \cdot \ln \mathsf{P}\{\|X^{(\nu)}\|_\rho \le \varepsilon\} =$$

$$= -\left(\frac{2\Gamma(\nu)}{\pi^{2\nu-1/2}\Gamma(\nu-1/2)}\right)^{1/(2\nu-1)} \frac{2\nu-1}{2} \left(\frac{\pi J_{2\nu}}{2\nu\sin(\pi/(2\nu))}\right)^{2\nu/(2\nu-1)}.$$

Exact asymptotics: $n = 1$

$$G_1(t,s) = \exp(-|t-s|),$$

hence $X^{(1)}(t)$ is the Ornstein-Uhlenbeck process.
The eigenvalues of the covariance of the process $X^{(1)}(t)$ satisfy $\lambda_k = \mu_k^{-1}$, where μ_k are the eigenvalues of the boundary value problem (BVP)

$$\begin{cases} -u'' + u = 2\mu u \quad \text{on} \quad [0,1], \\ u'(0) - u(0) = 0, \\ u'(1) + u(1) = 0. \end{cases}$$

As $\varepsilon \to 0$, one has

$$\mathrm{P}\{\|X^{(1)}\| \le \varepsilon\} \sim \frac{4e^{1/2}}{\pi^{1/2}} \varepsilon^2 \exp\left(-\frac{1}{4}\varepsilon^{-2}\right)$$

(A. Nazarov, J. Math. Sci., 2003).

Exact asymptotics: $n = 2$

$$G_2(t,s) = (1 + |t-s|)\exp(-|t-s|).$$

The eigenvalues of the covariance of the process $X^{(2)}(t)$ satisfy $\lambda_k = \mu_k^{-1}$, where μ_k are the eigenvalues of the BVP

$$\begin{cases} u^{IV} - 2u'' + u = 4\mu u \quad \text{on} \quad [0,1], \\ u''(0) - 2u'(0) + u(0) = 0, \\ u'''(0) - 2u''(0) + u'(0) = 0, \\ u''(1) + 2u'(1) + u(1) = 0, \\ u'''(1) + 2u''(1) + u'(1) = 0. \end{cases}$$

As $\varepsilon \to 0$, one has

$$\mathsf{P}\{\|X^{(2)}\| \le \varepsilon\} \sim \frac{2^{29/6} e}{3^{1/2}\pi^{1/2}} \varepsilon^{5/3} \exp\left(-\frac{3}{2^{7/3}} \varepsilon^{-2/3}\right).$$

Exact asymptotics: $n \in \mathbb{N}$

Recently we obtained the exact asymptotics for the process $X^{(n)}$ with arbitrary $n \in \mathbb{N}$.
The eigenvalues of the covariance of the process $X^{(n)}(t)$ satisfy $\lambda_k = \mu_k^{-1}$, where μ_k are the eigenvalues of the BVP

$$\begin{cases} (-1)^n(\mathrm{D}^2 - \mathrm{I})^n u = c_n \mu u \quad \text{on} \quad [0,1], \\ \mathrm{D}^m(\mathrm{D}+\mathrm{I})^n u(1) = 0, \quad m = 0,1,\dots,n-1, \\ \mathrm{D}^m(\mathrm{D}-\mathrm{I})^n u(0) = 0, \quad m = 0,1,\dots,n-1, \end{cases}$$

and $c_n = \frac{2\pi^{1/2}\Gamma(n)}{\Gamma(n-1/2)}$.

Exact asymptotics: $n \in \mathbb{N}$

Let $\pm\zeta_j$, $j = 1, \dots, n$, be the complex roots of the equation

$$(\zeta^2 + 1)^n = c_n \mu.$$

μ is the eigenvalue of the BVP iff $\pm\zeta_j$, $j = 1, \dots, n$, are the roots of the characteristic equation

$$\frac{\det[A_1, \dots, A_n]}{\det V(\zeta_1, -\zeta_1, \dots, \zeta_n, -\zeta_n)} = 0,$$

where $V(\dots)$ is the Vandermonde determinant and

$$A_j = \begin{bmatrix} (1+i\zeta_j)^n e^{i\zeta_j} & (1-i\zeta_j)^n e^{-i\zeta_j} \\ i\zeta_j(1+i\zeta_j)^n e^{i\zeta_j} & -i\zeta_j(1-i\zeta_j)^n e^{-i\zeta_j} \\ \dots\dots & \dots\dots \\ (i\zeta_j)^{n-1}(1+i\zeta_j)^n e^{i\zeta_j} & (-i\zeta_j)^{n-1}(1-i\zeta_j)^n e^{-i\zeta_j} \\ (1-i\zeta_j)^n & (1+i\zeta_j)^n \\ i\zeta_j(1-i\zeta_j)^n & -i\zeta_j(1+i\zeta_j)^n \\ \dots\dots & \dots\dots \\ (i\zeta_j)^{n-1}(1-i\zeta_j)^n & (-i\zeta_j)^{n-1}(1+i\zeta_j)^n \end{bmatrix}.$$

Exact asymptotics: $n \in \mathbb{N}$

We have

$$c_n \mu_k = (\pi(k - (n+1)/2) + O(k^{-1}))^{2n},$$

and, by the comparison theorem,

$$\mathsf{P}\{\|X^{(n)}\| \le \varepsilon\} \sim$$
$$C_{\rm dist}\,\mathsf{P}\left\{\sum_{k=1}^{n} \xi_k^2 + \sum_{k=n+1}^{\infty} \left(\pi\left(k - \frac{n+1}{2}\right)\right)^{-2n} \xi_k^2 \le \frac{\varepsilon^2}{c_n}\right\},$$

where

$$C_{\rm dist}^2 = \prod_{k=1}^{n} c_n \mu_k \prod_{k=n+1}^{\infty} \frac{c_n \mu_k}{\left(\pi(k - \frac{n+1}{2})\right)^{2n}}.$$

We calculate the distortion constant applying theorems from complex analysis to the characteristic determinant.

Exact asymptotics: $n \in \mathbb{N}$

As $\varepsilon \to 0$, one has

$$\mathsf{P}\{\|X^{(n)}\| \leq \varepsilon\} \sim \frac{2^{(n^2+n+1)/2} n^{(n+1)/2} e^{n/2}}{|\det V(1, z, \dots, z^{n-1})|} \frac{\varepsilon_n^{n^2+1}}{\sqrt{\pi \mathcal{D}_n}} \exp\left(-\frac{\mathcal{D}_n}{2\varepsilon_n^2}\right),$$

where

$$z = \exp\left(\frac{i\pi}{n}\right), \quad \mathcal{D}_n = \frac{2n-1}{2n \sin \frac{\pi}{2n}}, \quad \varepsilon_n = \left(\varepsilon \sqrt{\frac{2n}{c_n} \sin \frac{\pi}{2n}}\right)^{1/(2n-1)}.$$

Matérn fields

Matérn fields, i.e. tensor products of Matérn processes, find important applications in design and analysis of computer experiments (B. J. Williams, T. J. Santner, W. I. Notz, Statist. Sinica, 2000; W.-L. Loh, Ann. Statist., 2005).

Denote $\mathbb{X}_{(\nu_1,\dots,\nu_d)}(x) = \bigotimes_{j=1}^{d} X_j^{(\nu_j)}(x_j), x = (x_1, \dots, x_d) \in [0,1]^d$,

where $X_j^{(\nu_j)}$ are independent Matérn processes with parameter ν_j.

Tensor products

A. Karol', A. Nazarov, Y. Nikitin, Trans. Amer. Math. Soc., 2008:

If $\lambda_k(Y_j) \sim C_j k^{-p}$, $j = 1, \dots, d$, then

$$\lambda_k\Big(\bigotimes_{j=1}^{d} Y_j\Big) \sim \frac{\prod_{j=1}^{d} C_j}{(d-1)!^p} \cdot \frac{\log^{(d-1)p}(k+1)}{k^p}.$$

If $\lambda_k(Y) \sim Ck^{-p}$ and $\lambda_k(Y_j) = O(k^{-\tilde{p}})$, $j = 1, \dots, d$, with $\tilde{p} > p$, then

$$\lambda_k\Big(Y \bigotimes \Big(\bigotimes_{j=1}^{d} Y_j\Big)\Big) \sim \prod_{j=1}^{d} \left(\sum_{k=1}^{\infty} \lambda_k^{1/p}(Y_j)\right)^p \cdot Ck^{-p}.$$

Matérn fields

Let ρ be a summable nonnegative function on $[0,1]^d$. Then, for the field $\mathbb{X}_\nu = \mathbb{X}_{(\nu,\dots,\nu)}$, one has

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(2\nu-1)} \ln^{-2\nu(d-1)/(2\nu-1)}(1/\varepsilon) \cdot \ln \mathsf{P}\{\|\mathbb{X}_\nu\|_\rho \le \varepsilon\} =$$
$$= -\left(\frac{2\Gamma(\nu)}{\pi^{2\nu-1/2}\Gamma(\nu-1/2)}\right)^{d/(2\nu-1)} \cdot \left(\frac{2\nu-1}{2}\right)^{1-2\nu(d-1)/(2\nu-1)} \cdot$$
$$\cdot \left(\frac{\pi J_{2\nu}}{(d-1)! \cdot 2\nu \sin\frac{\pi}{2\nu}}\right)^{2\nu/(2\nu-1)}.$$

Matérn fields

Let $\nu_m = \min_j \nu_j$ and $\nu_j \neq \nu_m$ for $j \neq m$. Then

$$\lim_{\varepsilon \to 0} \varepsilon^{2/(2\nu_m - 1)} \cdot \ln \mathsf{P}\{\|\mathbb{X}_{(\nu_1,\dots,\nu_d)}\| \leq \varepsilon\} =$$

$$= -\prod_{j \neq m} \left(\sum_{n=1}^{\infty} (\lambda_n^{(j)})^{1/(2\nu_m)} \right)^{2\nu_m/(2\nu_m - 1)} \cdot$$

$$\cdot \left(\frac{2\Gamma(\nu_m)}{\pi^{2\nu_m - 1/2}\Gamma(\nu_m - 1/2)} \right)^{1/(2\nu_m - 1)} \cdot$$

$$\cdot \frac{2\nu_m - 1}{2} \cdot \left(\frac{\pi}{2\nu_m \sin \frac{\pi}{2\nu_m}} \right)^{2\nu_m/(2\nu_m - 1)},$$

where $\lambda_n^{(j)}$, $n \in \mathbb{N}$, are the eigenvalues of the covariance function of the process $X^{(\nu_j)}$.

Thank you for your attention!